Switched Linear Systems on Time Scales with Relaxed Commutativity Constraints

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***Abstract*— A fundamental stability result for arbitrarily switched linear systems in continuous time assumes that the set of system coefficient matrices are commutative with one another. This result was recently generalized to include arbitrarily switched linear system on arbitrary time scales $\mathbb{T}$, with additional constraints imposed upon the graininess of the time scales. In the following analysis we explore the case when pairs of switched systems are non-commutative by visualizing the space of common Lyapunov solutions graphically. We deduce that there are cases in which a common Lyapunov solution exists for a non-commutative switched system if the time scale graininess is limited to some upper bound[1].**

I. Introduction

Dynamical systems modeled as mixtures of discrete-event switching logic and standard difference or differential equations often belong to a class termed "switched systems." Typical examples of switched systems are vehicle transmissions, in which the vehicle dynamics change essentially instantaneously through gear shifts, or biological systems in which cell regulatory dynamics change suddenly depending on protein concentration levels. Two excellent overviews are given in the references [15],[17].

Another relevant example is the distributed control network, in which closed-loop controllers share congested communication networks that connect sensors nodes, actuators and other controllers. This example is particularly interesting, because the nature of the communication channel (the "network") may determine not only the characteristic switching modes but also the timing of the system. In other words, the underlying time domain – the times at which communication packets are transmitted or received – is neither continuous nor uniformly discrete as is usually expected or assumed; neither the continuous real line $\mathbb{R}$ nor the integers $\mathbb{Z}$ appropriately capture the temporal nature of the system [7],[8],[12].

To meet the challenge of switched systems with variable time domains, we employ the field of dynamic equations on time scales (DETS). An introduction is given in the appendix, but briefly a *time scale* $\mathbb{T}$ is any closed subset of $\mathbb{R}$, and the time scale *graininess* $\mu(t)$ refers to the distance from one point t in $\mathbb{T}$ to the next. Tools and results from the field of DETS allow dynamical systems to be modeled and analyzed on virtually any time scale through the use of generalized differential equations [2]. Not surprisingly, as $\mu \to 0$, time scale dynamic equations reduce to standard differential equations; as $\mu \to 1$ they reduce to standard difference equations. Time scales that are discrete (no continuous subintervals) with non-uniform step sizes naturally fit the problems of networked, distributed systems.

In the next section, we set up the time scale switched system stability problem, and briefly highlight some results that give sufficient conditions for stability. The main contribution of the paper then follows, a discussion of the geometry of the switched system stability problem when system commutativity constraints are relaxed.

II. Switched Systems on Time Scales

Let $\mathcal{A} := \{A_1, A_2, ..., A_m\}$ be a set of m matrices in $\mathbb{R}^{n\times n}$ with non-repeated eigenvalues, and $s : \mathbb{T} \to \{1, 2, ..., m\}$ be a switching signal, where $\mathbb{T}$ is a time scale. The switched linear system

$$x^{\Delta}(t) = A_{s(t)}x(t),\ t \geq 0,\ x(0) = x_0,\ t \in \mathbb{T}, \qquad (1)$$

has unique solution $x : \mathbb{T} \to \mathbb{R}^n$. Throughout the ensuing discussion, we make the following assumptions unless otherwise noted:

- A1 Switching signal s is arbitrary over $\mathbb{T}$. (This gives rise to the "arbitrary" switched system problem.)
- A2 For each $t \in \mathbb{T}$All eigenvalues of $A_i \in \mathcal{A}$ lie strictly within the Hilger circle. (In other words, each individual system is asymptotically stable, meaning that $x^{\Delta}(t) = A_i x(t)$ has $\|x(t)\| \to 0$ for as $t \to \infty$.)
- A3 Each A_i is regressive for all $t \in \mathbb{T}$.
- A4 All elements of $\mathcal{A}$ commute pair-wise, i.e. $A_iA_j - A_jA_i = 0$ for all $A_i, A_j \in \mathcal{A}$.
- A5 $\mathbb{T}$ has the following properties: (i) $0 \in \mathbb{T}$, (ii) $\mathbb{T}$ is unbounded above, and (iii) $\mathbb{T}$ has graininess $0 \leq \mu(t) \leq \mu_{\max}$ for all $t \in \mathbb{T}$. (At most, $\mu_{\max}$ must be selected so that A2 remains valid.)

Without loss of generality, some of the discussion that follows is restricted to the case with two switched systems to preserve clarity. Also, we assume that all quantities except A_i are time-varying unless otherwise noted. To examine the stability of (1), we propose the Lyapunov candidate

$$V = x^T P x > 0, \qquad (2)$$

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and examine the sufficient condition for stability (in the sense of Lyapunov) imposed by Lyapunov's second method:

$$A_i^T P + PA_i + \mu A_i^T PA_i + (I + \mu A_i^T)P^{\Delta}(I + \mu A_i) < 0. \quad (3)$$

The equation above, known as the Time Scale Dynamic Lyapunov Equation [4], illustrates one of the essential problems of switched system stability analysis: finding a "common" function (or, equivalently in this case, a common Lyapunov solution P) that applies for all i. There is no *a priori* guarantee that a common solution exists for any two systems A_i – even when each individual system is stable over the entire time scale! However, on $\mathbb{T} = \mathbb{R}$ it is known that assumption A4 gives a sufficient condition for the existence of a common quadratic Lyapunov function [15]; the same result was later extended to include switched systems on any $\mathbb{T}$ that meets the conditions of A5 [16] and an additional constraint imposed upon the graininess $\mu(t)$.

The matter of the additional constraint is discussed elsewhere at length [14],[16]. In short, there exists a region $\Re \subset \mathbb{R}^2$ such that common solutions to (3) exist when $\{\mu^\sigma, \mu\} \in \Re$. Imposing $\{\mu^\sigma, \mu\} \in \Re$ for the entire time scale guarantees that

$$(I + \mu A_i)^T P^{\Delta}(I + \mu A_i) - M_i < 0. \quad (4)$$

for some $M_i = M_i^T > 0$, and hence, solving

$$A_i^T P + PA_i + \mu A_i^T PA_i = -M_i \quad (5)$$

for all i is equivalent to solving (3). Note that (5) is termed the Time Scale Algebraic Lyapunov Equation (TSALE). The two-dimensional region $\Re$ is defined as the area under $2mn$ constraint curves, whose closed form is known and derived in the cited works. An example, for two systems of $n = 2$, is given in Figure 1. The upshot of the graininess constraint is

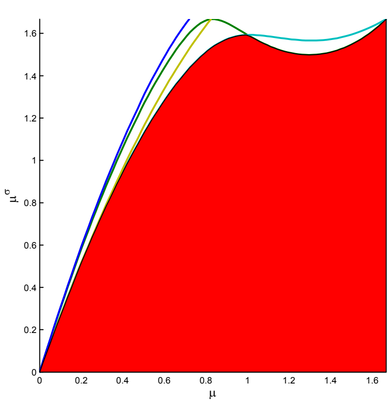

Fig. 1. Region $\Re$ is shaded above for two systems with eigenvalues $\{-1.2, -0.8\}$ and $\{-1 \pm 0.2j\}$. Note that 4 of the 8 constraint curves appear in the plot; the other 4 are outside of the plot boundary.

that, while it is always possible to preserve stability while "downshifting" the graininess (letting $\mu^\sigma < \mu$), arbitrary "upshifting" is not permitted. However, we stress that the graininess upshifting constraint is one of several *sufficient* conditions for the stability of an arbitrarily switched linear system.

III. The Commutativity Constraint

Another sufficient condition, the real "elephant in the room," is the system coefficient commutativity constraint imposed by A4, which is arguably a far more restrictive condition than the graininess upshift constraint. In point of fact, more general results do exist that can replace A4 (for $\mathbb{T} = \mathbb{R}$). One noteworthy result [15] guarantees the existence of a common Lyapunov function if any convex combination of A_i and A_j is itself a stable system; however the result only applies to families of systems with $n = 2$. Another, stronger, result gives the existence of a common Lyapunov function if *and only if* the Lie algebra generated by the set $\mathcal{A}$ is solvable, with the standard Lie bracket commutator. The drawback to this seemingly strongest of guarantees is that it is very difficult or impossible to test, practically speaking.

To attempt to understand what the commutativity condition really means, it is useful to visualize the problem. This is more easily done for $m = 2$ switched systems of dimension $n = 2$. In this case we have two simultaneous TSDLEs, written as

$$A_1^T P + PA_1 + \mu A_1^T PA_1 \quad < \quad 0, \quad (6)$$
$$A_2^T P + PA_2 + \mu A_2^T PA_2 \quad < \quad 0. \quad (7)$$

Since it is required by definition that $P = P^T > 0$, the solution matrix P is isomorphic to $\mathbb{R}^3$ and it can be viewed as a "point" in 3-space. Let $\mathcal{P}_1 \subset \mathbb{R}^3$ be the set of all solutions to (6), and $\mathcal{P}_2 \subset \mathbb{R}^3$ be the set of all solutions to (7). It is straightforward to show that $\mathcal{P}_1$ and $\mathcal{P}_2$ are convex sets; plotted in 3-space, they assume the shape of conic sections whose "tips" touch (but exclude) the origin.

Thus, the essential problem of simply knowing whether a common quadratic Lyapunov function *exists* boils down to knowing whether $\mathcal{P}_1 \cap \mathcal{P}_2 \neq \varnothing$. This paper does not address that question directly (readers are referred to a parallel SSST 2011 paper [6] for a discussion about knowing whether a common Lyapunov function *does not* exist). However, the geometry is revealing and suggests some interesting possible avenues of exploration.

To begin, consider the case when condition A4 is in force. An iterative method for finding P, seen in [15] and recently adapted for time scales by Miller, Ramos, *et. al.*, can be geometrically interpreted as: (A) Find, iteratively, solution P within the smallest $\mathcal{P}$ (i.e. the cone with the smallest cross-sectional area, we'll call this $\mathcal{P}_m$); (B) use A4 to show that $\mathcal{P}_i \subseteq \mathcal{P}_m$ for all $i < m$ and therefore $P \in \mathcal{P}_i$. This is nicely illustrated in Figure 5. Equally interesting, though, are cases where A_1 and A_2 become progressively "less" commutative. Although commutativity is a yes/no property (numerics notwithstanding), there is a continuum of "closeness" to commutativity that appears to correlate with the relative size of the intersection $\mathcal{P}_1 \cap \mathcal{P}_2$. This is illustrated in the sequence of Figures 2 through 5; the first figure showing a case where no common solution exists.

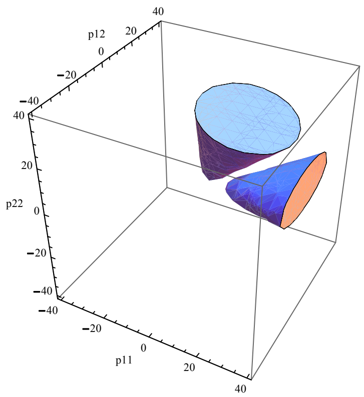

Fig. 2. The $\mathcal{P}_1$ and $\mathcal{P}_2$ cones, for two matrices A_1 and A_2 that do not commute and do not admit a common quadratic L yapunov solution.

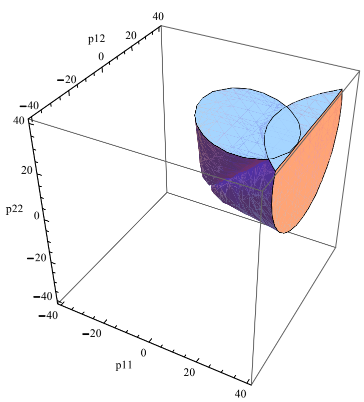

Fig. 3. Compared to the previous figure, A_2 modified so that it does not commute with A_1 but admits a commom solution, i.e. $\mathcal{P}_1 \cap \mathcal{P}_2$ is non-empty.

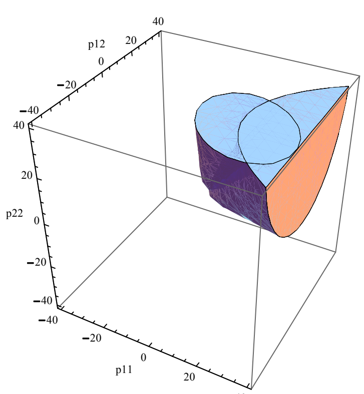

Fig. 4. $\mathcal{P}_1 \cap \mathcal{P}_2$ is growing as the coefficient matrices move "nearer" to commutativity.

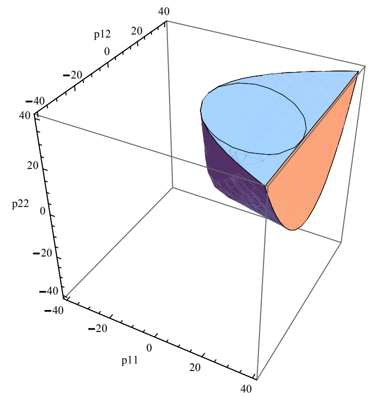

Fig. 5. A_1 and A_2 are commutative, and $\mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P}_1$.

The preceding figures and discussion assumed, without loss of generality, that $\mu = 0$ in (6) and (7), thus reverting to the case of the standard algebraic Lyapunov equation. However, on a general time scale $\mathbb{T}$ the variable graininess requires a new solution to (6) and (7) at every point in time. This gives rise to the question, how does the graininess impact the existence and size of $\mathcal{P}_1 \cap \mathcal{P}_2$? Intuitively, the negative-definiteness of a TSALE comes from the first two terms. The positivity and symmetry of the third term, $\mu A_i^T P A_i$, suggests that, as μ increases, the space $\mathcal{P}_i$ must shrink (i.e. the cone will get narrower but its central axis will not change direction). Therefore, if $\mathcal{P}_1 \cap \mathcal{P}_2$ was non-empty for small graininess, it may very well become empty above some critical graininess we term μ_{crit}. Figures 6 - 9 illustrate this idea.

IV. Conclusion

This paper is focused on the challenge of proving the stability of two or more arbitrarily switched linear systems on time scales using common quadratic Lyapunov functions, even when the system matrices are non-commutative. A problem statement was given, followed by a brief summary of the nature of, and constraints upon, common solutions in the case of commutative system matrices. Then a geometric view of the problem was presented.

We are left with two intriguing questions:

1) Is there a metric that can be applied to the matrix commutator that will indicate when $\mathcal{P}_i \cap \mathcal{P}_j = \varnothing$ for $\mu = 0$? Even if such a metric were conservative (i.e. it provided a testable sufficient condition) it would still be quite useful.
2) As evidenced by the last example, if a common solution exists for $\mu = 0$, there is a hard upper bound μ_{crit} on the graininess such that no common solutions exists when $\mu \geq \mu_{crit}$. Can μ_{crit} be predicted given the A_i matrices, and when is $\mu_{crit} \leq \mu_{\max}$? (It may be that the critical graininess is sometimes larger than the maximum graininess that will keep A5 in force.)

While the preceding discussion involves switched systems with arbitrary switching on an arbitrary (upper bounded) time scale, there is also the open question of constrained switching

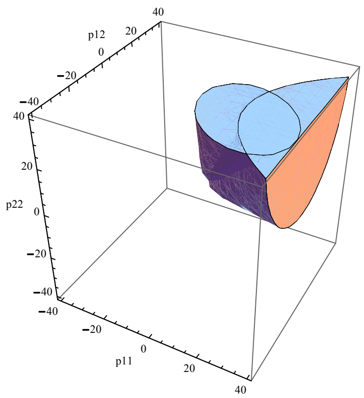

Fig. 6. A TSALE with $\mu = 0$ is equavalent to a standard Lyapunov equation. Here, A_1 and A_2 do not commute but do admit a common quadratic Lyapunov solution.

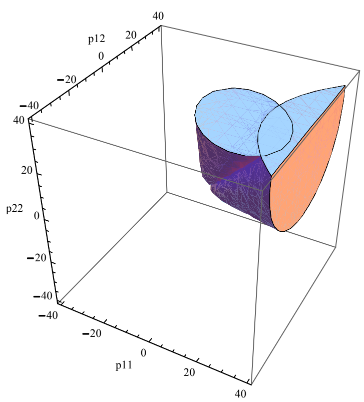

Fig. 7. The same A matrices, but increaseing the graininess to $\mu = 0.5$.

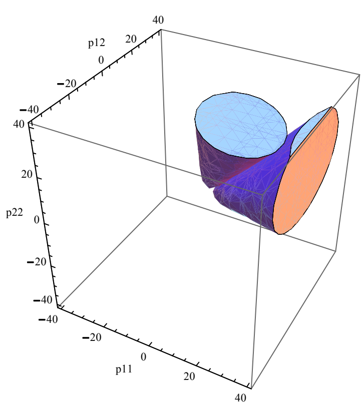

Fig. 8. The critical graininess is $\mu_{crit} = 0.9$.

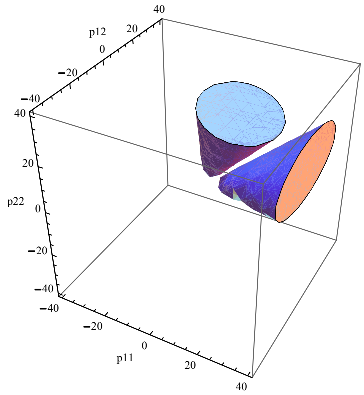

Fig. 9. For graininess $\mu > \mu_{crit}$, no common solutions are admitted.

stability on time scales, in which the choice of each successive system matrix A_i is not arbitrary but based upon knowledge of the state or some other information. On $\mathbb{T} = \mathbb{R}$, it is known that constrained switching can results in a stable switched system whose "ingredient" systems are themselves unstable.

V. Appendix: Dynamic Equations on Time Scales

A. What Are Time Scales?

This appendix is reproduced from the authors' previous works as convenience to readers not yet familiar with the theory of time scales [5]. The theory of time scales springs from the 1988 doctoral dissertation of Stefan Hilger [11] that resulted in his seminal paper [10]. These works aimed to unify various overarching concepts from the (sometimes disparate) theories of discrete and continuous dynamical systems [13], but also to extend these theories to more general classes of dynamical systems. From there, time scales theory advanced fairly quickly, culminating in the excellent introductory text by Bohner and Peterson [3] and the more advanced monograph [2]. A succinct survey on time scales can be found in [1].

A *time scale* $\mathbb{T}$ is any non-empty, (topologically) closed subset of the real numbers $\mathbb{R}$. Thus time scales can be (but are not limited to) any of the usual integer subsets (e.g. $\mathbb{Z}$ or $\mathbb{N}$), the entire real line $\mathbb{R}$, or any combination of discrete points unioned with closed intervals. For example, if $q > 1$ is fixed, the *quantum time scale* $\overline{q^{\mathbb{Z}}}$ is defined as

$$\overline{q^{\mathbb{Z}}} := \{q^k : k \in \mathbb{Z}\} \cup \{0\}.$$

The quantum time scale appears throughout the mathematical physics literature, where the dynamical systems of interest are the q-difference equations. Another interesting example is the *pulse time scale* $\mathbb{P}_{a,b}$ formed by a union of closed intervals each of length a and gap b:

$$\mathbb{P}_{a,b} := \bigcup_k \left[k(a+b), k(a+b)+a\right].$$

Other examples of interesting time scales include any collection of discrete points sampled from a probability distribution, any sequence of partial sums from a series with positive terms, or even the famous Cantor set.

TABLE I
DIFFERENTIAL OPERATORS ON TIME SCALES.

time scale	differential operator	notes	integral operator	notes
$\mathbb{T}$	$x^{\Delta}(t) = \frac{x(\sigma(t))-x(t)}{\mu(t)}$	generalized derivative	$\int_a^b f(t)\,\Delta t$	generalized integral
$\mathbb{R}$	$x^{\Delta}(t) = \lim_{h\to 0} \frac{x(t+h)-x(t)}{h}$	standard derivative	$\int_a^b f(t)\Delta t = \int_a^b f(t)\,dt$	standard Lebesgue integral
$\mathbb{Z}$	$x^{\Delta}(t) = \Delta x(t) := x(t+1)-x(t)$	forward difference	$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$	summation operator
$h\mathbb{Z}$	$x^{\Delta}(t) = \Delta_h x(t) := \frac{x(t+h)-x(t)}{h}$	h-forward difference	$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-h} f(t)h$	h-summation
$\overline{q^{\mathbb{Z}}}$	$x^{\Delta}(t) = \Delta_q x(t) := \frac{x(qt)-x(t)}{(q-1)t}$	q-difference	$\int_a^b f(t)\Delta t = \sum_{t=a}^{b/q} \frac{f(t)}{(q-1)t}$	q-summation
$\mathbb{P}_{a,b}$	$x^{\Delta}(t) = \begin{cases} \frac{dx}{dt}, & \sigma(t)=t, \\ \frac{x(t+b)-x(t)}{b}, & \sigma(t)>t \end{cases}$	pulse derivative	$\int_I f(t)\Delta t = \begin{cases} \int_I f(t)dt, & \sigma(t)=t \\ f(t)\mu(t), & \sigma(t)>t \end{cases}$	pulse integral

The bulk of engineering systems theory to date rests on two time scales, $\mathbb{R}$ and $\mathbb{Z}$ (or more generally $h\mathbb{Z}$, meaning discrete points separated by distance h). However, there are occasions when necessity or convenience dictates the use of an alternate time scale. The question of how to approach the study of dynamical systems on time scales then becomes relevant, and in fact the majority of research on time scales so far has focused on expanding and generalizing the vast suite of tools available to the differential and difference equation theorist. We now briefly outline the portions of the time scales theory that are needed for this paper to be as self-contained as is practically possible.

B. The Time Scales Calculus

The *forward jump operator* is given by $\sigma(t) := \inf_{s\in\mathbb{T}}\{s > t\}$, while the *backward jump operator* is $\rho(t) := \sup_{s\in\mathbb{T}}\{s < t\}$. The *graininess function* $\mu(t)$ is given by $\mu(t) := \sigma(t) - t$.

A point $t \in \mathbb{T}$ is *right-scattered* if $\sigma(t) > t$ and *right dense* if $\sigma(t) = t$. A point $t \in \mathbb{T}$ is *left-scattered* if $\rho(t) < t$ and *left dense* if $\rho(t) = t$. If t is both left-scattered and right-scattered, we say t is *isolated* or *discrete*. If t is both left-dense and right-dense, we say t is *dense*. The set $\mathbb{T}^{\kappa}$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$, define $f^{\Delta}(t)$ as the number (when it exists), with the property that, for any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\big|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\big| \leq \epsilon|\sigma(t) - s|, \quad \forall s \in U. \tag{8}$$

The function $f^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$ is called the *delta derivative* or the *Hilger derivative* of f on $\mathbb{T}^{\kappa}$. Equivalently, (8) can be restated to define the Δ-differential operator as

$$x^{\Delta}(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)},$$

where the quotient is taken in the sense that $\mu(t) \to 0^+$ when $\mu(t) = 0$.

A benefit of this general approach is that the realms of differential equations and difference equations can now be viewed as but special, particular cases of more general *dynamic equations on time scales*, i.e. equations involving the delta derivative(s) of some unknown function. See Table **??**.

Naturally, with any discussion of derivatives a notion of "continuity" is required. For $f : \mathbb{T} \to \mathbb{X}$, the function f is said to be right-dense continuous, or *rd-continuous*, if it is continuous (in the usual sense) over any right-dense interval within $\mathbb{T}$. The set of all rd-continuous functions that are n-times differentiable is denoted $C^n_{rd}(\mathbb{T}, \mathbb{X})$.

Since the graininess function induces a measure on $\mathbb{T}$, if we consider the Lebesgue integral over $\mathbb{T}$ with respect to the μ-induced measure,

$$\int_{\mathbb{T}} f(t)\, d\mu(t),$$

then all of the standard results from measure theory are available [9]. The upshot is that the derivative and integral concepts apply just as readily to *any* closed subset of the real line as they do on $\mathbb{R}$ or $\mathbb{Z}$; see Table 1. Our goal is to leverage this general framework against wide classes of dynamical and control systems.

The function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$. We define the related sets

$$\mathcal{R} := \{p : \mathbb{T} \to \mathbb{R} : p \in C_{rd}(\mathbb{T}) \text{ and } 1 + \mu(t)p(t) \neq 0 \text{ for all } t \in \mathbb{T}^{\kappa}\},$$
$$\mathcal{R}^+ := \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}^{\kappa}\}.$$

For $p(t) \in \mathcal{R}$, we define the *generalized time scale exponential function* $e_p(t, t_0)$ as the unique solution to the initial value problem $x^{\Delta}(t) = p(t)x(t)$, $x(t_0) = 1$, which exists when $p \in \mathcal{R}$. See [2]. The system eigenvalue, $p(t)$, is said to belong to the *Hilger Circle* when $|1 + p(t)\mu(t)| < 1$.

Similarly, the unique solution to the matrix initial value problem $X^{\Delta}(t) = A(t)X(t)$, $X(t_0) = I$ is called the *transition matrix* associated with this system. This solution is denoted by $\Phi_A(t, t_0)$ and exists when $A \in \mathcal{R}$. A matrix is regressive if and only if all of its eigenvalues are in $\mathcal{R}$. Equivalently, the matrix $A(t)$ is regressive if and only if $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^{\kappa}$.

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