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Existence results for singular three point boundary value problems on time scales

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Abstract

We prove the existence of a positive solution for the three point boundary value problem on time scale $\mathbb{T}$ given by

$$y^{\Delta\Delta} + f(x, y) = 0, \quad x \in (0, 1] \cap \mathbb{T}, \quad y(0) = 0, \quad y(p) = y\big(\sigma^2(1)\big),$$

where $p \in (0, 1) \cap \mathbb{T}$ is fixed and $f(x, y)$ is singular at $y = 0$ and possibly at $x = 0$, $y = \infty$. We do so by applying a fixed point theorem due to Gatica, Oliker, and Waltman [J. Differential Equations 79 (1989) 62] for mappings that are decreasing with respect to a cone. We also prove the analogous existence results for the related dynamic equations $y^{\nabla\nabla} + f(x, y) = 0$, $y^{\Delta\nabla} + f(x, y) = 0$, and $y^{\nabla\Delta} + f(x, y) = 0$ satisfying similar three point boundary conditions.

Keywords: Singular boundary value problem; Time scale; Fixed point theorem

1. Introduction

We are interested in the existence of a positive solution for the three point boundary value problem on a time scale $\mathbb{T}$,

$$y^{\Delta\Delta} + f(x, y) = 0, \quad x \in (0, 1]_{\mathbb{T}}, \tag{1.1}$$

$$y(0) = 0, \quad y(p) = y\big(\sigma^2(1)\big), \tag{1.2}$$

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where $f(x, y)$ is singular at $y = 0$, and possibly at $x = 0$ and $y = \infty$. (An interval subscripted with $\mathbb{T}$ is the one intersected with $\mathbb{T}$.) We make the following assumptions:

(A1) $p \in (0, 1)_{\mathbb{T}}$ is arbitrary but fixed; $0, 1 \in \mathbb{T}$ with 0 right dense;
(A2) $f : (0, 1]_{\mathbb{T}} \times (0, \infty)_{\mathbb{T}} \to (0, \infty)$ is decreasing in y for every $x \in (0, 1]_{\mathbb{T}}$;
(A3) $\lim_{y\to 0^+} f(x, y) = \infty$ and $\lim_{y\to\infty} f(x, y) = 0$, uniformly on compact subsets of $(0, \sigma^2(1)]_{\mathbb{T}}$;
(A4) $0 < \int_0^{\sigma^2(1)} f(x, g_\theta(x))\Delta x < \infty$ for all $\theta > 0$ and g_θ as defined in (2.1).

To make this work reasonably self-contained we have included the basic definitions from the theory of time scales in Appendix A.

The seminal paper by Gatica, Oliker, and Waltman [21] in 1989 has had a profound impact on the study of singular boundary value problems for ordinary differential equations (ODEs). They studied singularities of the type in (A2)–(A4) for second order Sturm–Louiville problems, and their key result hinged on an application of a particular fixed point theorem for operators which are decreasing with respect to a cone. Various authors have used these techniques to study singular problems of various types. For example, Henderson and Yin [24,25] as well as Eloe and Henderson [15–17] have studied right focal, focal, conjugate, and multipoint singular boundary value problems for ODEs. Baxley [12], Erbe and Kong [19], and Fink, Gatica, and Hernández [20] are also excellent references which make use of [21]. For completeness, we do note that there are papers which deal with singular problems of this type without appealing to the results of [21]; for example, see [28].

However, the time scale setting here is much more general since ODEs and finite difference equations are but special cases of the *dynamic equation* given by (1.1). This is a rapidly expanding area of research; we refer the reader to the excellent introductory text by Bohner and Peterson [14] as well as their recent research monograph [13]. Problems such as (1.1) are dealt with quite extensively in [7] as well as in [1,13,14]. The paper by Eloe, Sheng, and Henderson [18] is a very interesting study in the numerical aspects of these equations. We note that this is the first work (to our knowledge) that deals with singular boundary value problems in a general time scales setting.

In particular, three point boundary conditions such as (1.2) have been investigated for the continuous case (ODEs), the discrete cases (difference equations), and the general time scales case by Anderson [3–5], Gupta [22,23], and Ma [29], to name a few. Very recently, Singh [30] established the existence of a positive solution to (1.1), (1.2) in the special case $\mathbb{T} = \mathbb{R}$ by using the methods of [21]; certainly [30] is the motivation for this paper.

We have organized the paper as follows. In Section 2, we start with some preliminary definitions and results from the study of cones in Banach spaces and state an important fixed point theorem from [21]. We formulate two lemmas which establish a priori upper and lower bounds on solutions of (1.1), (1.2). We then state and prove our main existence theorem. In Section 3, we consider the so-called "nabla–nabla" problem analogous to (1.1) satisfying similar boundary conditions and singularity assumptions on the nonhomogeneity. In Section 4, we consider the "mixed" dynamic equations of "delta–nabla" and "nabla–delta" type with boundary conditions similar to (1.2). The Green's functions for all four problems here are new. For the convenience of the reader, we conclude with a very brief appendix which should serve as a time scales primer for those unfamiliar with the area.

2. The delta–delta problem

We begin by giving definitions and some properties of cones in a Banach space. For references, see Krasnosel'skii [27] and Amann [2].

Let $\mathcal{B}$ be a real Banach space. A nonempty set $\mathcal{K} \subset \mathcal{B}$ is called a *cone* if the following conditions are satisfied:

(a) the set $\mathcal{K}$ is closed;
(b) if $u, v \in \mathcal{K}$ then $\alpha u + \beta v \in \mathcal{K}$ for all real $\alpha, \beta \geqslant 0$;
(c) $u, -u \in \mathcal{K}$ imply $u = 0$.

A cone $\mathcal{K}$ is *normal* in $\mathcal{B}$ provided there exists $\delta > 0$ such that $\|e_1 + e_2\| \geqslant \delta$, for all $e_1, e_2 \in \mathcal{K}$ with $\|e_1\| = \|e_2\| = 1$. Given a cone $\mathcal{K}$ a *partial order*, $\preccurlyeq$, is induced on $\mathcal{B}$ by $x \preccurlyeq y$, for $x, y \in \mathcal{B}$ if and only if $y - x \in \mathcal{K}$. For clarity, we sometimes write $x \preccurlyeq y$ (w.r.t. $\mathcal{K}$). If $x, y \in \mathcal{B}$ with $x \preccurlyeq y$, let $\langle x, y\rangle$ denote the *closed order interval between x and y* given by $\langle x, y\rangle = \{z \in \mathcal{B} \mid x \preccurlyeq z \preccurlyeq y\}$.

The following result due to Krasnosel'skii will be needed later.

Theorem 2.1 [27, p. 24]. *If $\mathcal{K} \subset \mathcal{B}$ is a normal cone, then closed order intervals are norm bounded.*

Next we state the fixed point theorem due to Gatica, Oliker, and Waltman [21] which is instrumental in proving our existence results.

Theorem 2.2 (Gatica–Oliker–Waltman fixed point theorem). *Let $\mathcal{B}$ be a Banach space, $\mathcal{K} \subset \mathcal{B}$ be a normal cone, and $D \subset \mathcal{K}$ be such that if $x, y \in D$ with $x \preccurlyeq y$, then $\langle x, y\rangle \subset D$. Let $T : D \to \mathcal{K}$ be a continuous, decreasing mapping which is compact on any closed order interval contained in D, and suppose there exists an $x_0 \in D$ such that T^2x_0 is defined (where $T^2x_0 = T(Tx_0)$) and Tx_0, T^2x_0 are order comparable to x_0. Then T has a fixed point in D provided that either*:

(i) *$Tx_0 \preccurlyeq x_0$ and $T^2x_0 \preccurlyeq x_0$;*
(ii) *$x_0 \preccurlyeq Tx_0$ and $x_0 \preccurlyeq T^2x_0$; or*
(iii) *The complete sequence of iterates $\{T^n x_0\}_{n=0}^{\infty}$ is defined and there exists $y_0 \in D$ such that $Ty_0 \in D$ with $y_0 \preccurlyeq T^n x_0$ for all $n \in \mathbb{N}$.*

We seek positive solutions, $y : [0, \sigma^2(1)]_{\mathbb{T}} \to \mathbb{R}^+$, satisfying (1.1), (1.2). To accomplish this, we transform (1.1), (1.2) into an integral equation involving the appropriate Green's function, and seek fixed points of the underlying integral operator. We will then show that these fixed points form a sequence of iterates converging to a solution of (1.1), (1.2).

First, note that positive solutions of (1.1), (1.2) are also concave on $[0, \sigma^2(1)]_{\mathbb{T}}$. Consider the Banach space, $\mathcal{B} = C[0, \sigma^2(1)]_{\mathbb{T}}$, with the norm

$$\|u\| = \sup_{x \in [0, \sigma^2(1)]_{\mathbb{T}}} \left|u(x)\right|.$$

Define the normal cone, $\mathcal{K} \subset \mathcal{B}$, via

$$\mathcal{K} := \left\{u \in \mathcal{B} \mid u(x) \geqslant 0 \text{ on } \left[0, \sigma^2(1)\right]_{\mathbb{T}}\right\},$$

and define the *tent function* $g_1 : [0, \sigma^2(1)]_{\mathbb{T}} \to [0, \infty)$ by

$$g_1(x) = \begin{cases} x, & \text{if } 0 \leqslant x \leqslant p, \\ p, & \text{if } \sigma(p) \leqslant x \leqslant \sigma^2(1). \end{cases}$$

Finally, for $\theta > 0$, let

$$g_\theta(x) = \theta \cdot g_1. \tag{2.1}$$

We observe that for each positive (and concave) solution, $y(x)$, of (1.1), (1.2), there exists some $\theta > 0$ such that $g_\theta(x) \leqslant y(x)$ for all $x \in [0, \sigma^2(1)]_{\mathbb{T}}$.

We will apply Theorem 2.2 to operators whose kernel is the Green's function for $-y^{\Delta\Delta} = 0$ and satisfies (1.2). The Green's function, $G : [0, \sigma^2(1)]_{\mathbb{T}} \times [0, \sigma(1)]_{\mathbb{T}} \to [0, \infty)$, is given by

$$G(x,t) = \begin{cases} x, & x \leqslant t \leqslant p, \\ \sigma(t), & \sigma(t) \leqslant x \text{ and } t \leqslant p, \\ \frac{\sigma^2(1) - \sigma(t)}{\sigma^2(1) - p} \cdot x, & \sigma(p) \leqslant t \text{ and } x \leqslant t, \\ \sigma(t) - x + \frac{\sigma^2(1) - \sigma(t)}{\sigma^2(1) - p} \cdot x, & \sigma(p) \leqslant t \leqslant \sigma(t) \leqslant x. \end{cases}$$

Notice that $G(x,t) > 0$ for $(x,t) \in (0, \sigma^2(1))_{\mathbb{T}} \times (0, \sigma(1))_{\mathbb{T}}$.

Define $D \subset \mathcal{K}$ by

$$D := \left\{\phi \in \mathcal{K} \mid \exists \theta(\phi) > 0 \text{ such that } \phi(x) \geqslant g_\theta(x),\ x \in \left[0, \sigma^2(1)\right]_{\mathbb{T}}\right\},$$

and the integral operator $T : D \to \mathcal{K}$ by

$$Tu(x) := \int_0^{\sigma^2(1)} G(x,t) f\bigl(t, u(t)\bigr) \Delta t.$$

It suffices to define D as above, since the singularity in f precludes us from defining T on all of $\mathcal{K}$. Furthermore, it can easily be verified that T is well-defined. In that direction, note that for $\phi \in D$ there exists $\theta(\phi) > 0$ such that $g_\theta(x) \leqslant \phi(x)$ for all $x \in [0, \sigma^2(1)]_{\mathbb{T}}$. Since $f(x, y)$ decreases with respect to y, we see $f(x, \phi(x)) \leqslant f(x, g_\theta(x))$ for $x \in (0, \sigma^2(1)]_{\mathbb{T}}$. Thus,

$$0 \leqslant \int_0^{\sigma^2(1)} G(x,t) f\bigl(t, \phi(t)\bigr) \Delta t \leqslant \int_0^{\sigma^2(1)} G(x,t) f\bigl(t, g_\theta(t)\bigr) \Delta t < \infty.$$

Similarly, T is decreasing with respect to D.

Lemma 2.1. *$\phi \in D$ is a solution of* (1.1), (1.2) *if and only if $T\phi = \phi$.*

Proof. One direction of the lemma is obviously true. To see the other direction, let $\phi \in D$. Then $(T\phi)(x) = \int_0^{\sigma^2(1)} G(x,t) f(t,\phi(t))\Delta t$, and $(T\phi)^{\Delta\Delta}(x) = -f(x,\phi(x)) < 0$ for $x \in (0,1]_{\mathbb{T}}$. Moreover, $(T\phi)(x) \geqslant 0$, $(T\phi)(0) = 0$, and $(T\phi)(p) - (T\phi)(1) = 0$. Thus, there exists some $\theta(T\phi)$ such that $(T\phi)(x) \geqslant g_\theta(x)$, which implies that $T\phi \in D$. That is, $T: D \to D$.

We now present two lemmas that are required in order to apply Theorem 2.2. The first establishes an a priori upper bound on solutions, while the second establishes an a priori lower bound on solutions.

Lemma 2.2. *If f satisfies* (A1)–(A4)*, then there exists an $S > 0$ such that $\|\phi\| \leqslant S$ for any solution $\phi \in D$ of* (1.1), (1.2).

Proof. For the sake of contradiction, suppose that the conclusion is false. Then there exists a sequence, $\{\phi_n\}_{n=1}^{\infty}$, of solutions to (1.1), (1.2) such that $\phi_n(x) > 0$ for $x \in (0,\sigma^2(1)]_{\mathbb{T}}$, and $\|\phi_n\| \leqslant \|\phi_{n+1}\|$ with $\lim_{n\to\infty} \|\phi_n\| = \infty$. Note that for any solution ϕ of (1.1), we have $\phi^{\Delta\Delta}(x) = -f(x,\phi(x)) < 0$ on $(0,1]_{\mathbb{T}}$; that is, ϕ is concave. In particular, the graph of each ϕ_n is concave. Furthermore, we claim that the boundary conditions (1.2) and the concavity of ϕ_n yield $\phi_n(x) > p\phi_n(x_n) = p\|\phi_n\|$, for $x \in [p,\sigma^2(1)]_{\mathbb{T}}$, where $x_n \in (p,\sigma^2(1))_{\mathbb{T}}$ is the abscissa of the maximum value of the solution, $\phi_n(x)$. To see this, we consider the line segment joining $(0,0)$ and $(x_n,\phi_n(x_n))$, given by $\ell(x) = \|\phi_n\|x/x_n$, $x \in [0,x_n]_{\mathbb{T}}$. Thus, $\ell(p) = \|\phi_n\|p/x_n > p\|\phi_n\|$. Furthermore, $\ell(p) < \phi_n(p)$ and $\phi_n(x) \geqslant \phi_n(p)$, $x \in [p,\sigma^2(1)]_{\mathbb{T}}$. Thus

$$\phi_n(x) \geqslant \phi_n(p) > \ell(p) > p\|\phi_n\|, \quad x \in \bigl[p,\sigma^2(1)\bigr]_{\mathbb{T}}, \tag{2.2}$$

which implies

$$\phi_n(x) > p\phi_n(x_n), \quad x \in \bigl[p,\sigma^2(1)\bigr]_{\mathbb{T}},$$

and hence the claim.

Let $\theta = p\phi_{n_0}(x_{n_0}) = p\|\phi_{n_0}\|$. Then the line segment joining $(0,0)$ with (p,θ) and the line segment joining (p,ϕ) with $(1,\theta)$ must lie under the graph of ϕ_n for $n \geqslant n_0$. That is, $\phi_n(x) \geqslant g_\theta(x)$ *for* $x \in [0,\sigma^2(1)]_{\mathbb{T}}$. Thus, for $n \geqslant n_0$ and $x \in (0,\sigma^2(1)]$, we have

$$\phi_n(x) = T\phi_n(x) = \int_0^{\sigma^2(1)} G(x,t) f\bigl(t,\phi_n(t)\bigr)\Delta t \leqslant \int_0^{\sigma^2(1)} G(x,t) f\bigl(t,g_\theta(t)\bigr)\Delta t < \infty.$$

But this contradicts the assumption that $\|\phi_n\| \to \infty$ as $n \to \infty$. Hence, there exists an $S > 0$ such that $\|\phi\| \leqslant S$ for any solution $\phi \in D$ of (1.1), (1.2). □

Lemma 2.3. *If f satisfies* (A1)–(A4)*, then there exists an $R > 0$ such that $\|\phi\| \geqslant R$ for any solution $\phi \in D$ of* (1.1), (1.2).

Proof. For the sake of contradiction, suppose $\phi_n(x) \to 0$ uniformly on $[0, \sigma^2(1)]_{\mathbb{T}}$ as $n \to \infty$. Let

$$m = \inf\{G(x,t)\colon (x,t) \in [p, \sigma^2(1)]_{\mathbb{T}} \times [p, \sigma(1)]_{\mathbb{T}}\} > 0.$$

From (A2), we see that $\lim_{y\to 0^+} f(x,y) = \infty$ uniformly on compact subsets of $(0, \sigma^2(1)]_{\mathbb{T}}$. Hence, there exists some $\delta > 0$ such that for $x \in [p, \sigma^2(1)]_{\mathbb{T}}$ and $0 < y < \delta$, we have $f(x,y) \geqslant 1/(m(1-p))$. On the other hand, there exists an $n_0 \in \mathbb{N}$ such that $n \geqslant n_0$ implies $0 < \phi_n(x) < \delta/2$, for $x \in (0, \sigma^2(1)]_{\mathbb{T}}$. So, for $x \in (p, \sigma^2(1)]_{\mathbb{T}}$ and $n \geqslant n_0$,

$$\phi_n(x) = T\phi_n(x) = \int_0^{\sigma^2(1)} G(x,t) f\bigl(t, \phi_n(t)\bigr)\Delta t \geqslant m \int_p^{\sigma^2(1)} f\bigl(t, \phi_n(t)\bigr)\Delta t$$

$$> m \int_p^{\sigma^2(1)} f(t, \delta/2)\Delta t \geqslant m \int_p^{\sigma^2(1)} \frac{1}{m(1-p)}\Delta t = 1.$$

But this contradicts the assumption that $\|\phi_n\| \to 0$ uniformly on $[0, \sigma^2(1)]_{\mathbb{T}}$ as $n \to \infty$. Hence, there exists an $R > 0$ such that $R \leqslant \|\phi\|$. $\square$

We now present the main result of the paper.

Theorem 2.3. *If f satisfies* (A1)–(A4), *then* (1.1), (1.2) *has at least one positive solution.*

Proof. For each $n \in \mathbb{N}$, let $\psi_n(x) = T(n)$, where n is the constant function of that value on $[0, \sigma^2(1)]_{\mathbb{T}}$. In particular,

$$\psi_n(x) = \int_0^{\sigma^2(1)} G(x,t) f(t,n)\Delta t.$$

Since f is decreasing in its second component and T is also a decreasing mapping,

$$\psi_{n+1}(x) \leqslant \psi_n(x), \quad \psi_n(x) > 0, \quad x \in \bigl(0, \sigma^2(1)\bigr]_{\mathbb{T}}. \tag{2.3}$$

By (A2), $\psi_n(x) \to 0$ uniformly on $[0, \sigma^2(1)]_{\mathbb{T}}$ as $n \to \infty$. Define $f_n : (0,1]_{\mathbb{T}} \times [0,\infty)_{\mathbb{T}} \to (0,\infty)$ by

$$f_n(x,t) = f\bigl(x, \max\{t, \psi_n(x)\}\bigr).$$

Note that f_n has effectively "removed the singularity" in f at $y = 0$. Moreover, for $(x,t) \in (0,1]_{\mathbb{T}} \times (0,\infty)_{\mathbb{T}}$, we see $f_n(x,t) \leqslant f(x,t)$, and in particular,

$$f_n(x,t) = f\bigl(x, \max\{t, \psi_n(x)\}\bigr) \leqslant f\bigl(x, \psi_n(x)\bigr).$$

Next, define a sequence of operators $T_n : \mathcal{K} \to \mathcal{K}$ via

$$T_n\phi(x) := \int_0^{\sigma^2(1)} G(x,t) f_n\bigl(t, \phi(t)\bigr)\Delta t, \quad \phi \in \mathcal{K},\ x \in \bigl(0, \sigma^2(1)\bigr]_{\mathbb{T}}.$$

From standard arguments involving the Arzelà–Ascoli Theorem we know that each T_n is in fact a compact mapping on $\mathcal{K}$. Furthermore, $T_n(0) \geqslant 0$ and $T_n^2(0) \geqslant 0$. By Theorem 2.2, for each $n \in \mathbb{N}$, there exists $\phi_n \in \mathcal{K}$ such that $T_n\phi_n(x) = \phi_n(x)$ for $x \in [0, \sigma^2(1)]_{\mathbb{T}}$. Hence, for each $n \in \mathbb{N}$, ϕ_n satisfies the boundary conditions of the problem. In addition, for each ϕ_n,

$$T_n\phi_n(x) = \int_0^{\sigma^2(1)} G(x,t) f_n\bigl(t, \phi_n(t)\bigr)\Delta t = \int_0^{\sigma^2(1)} G(x,t) f\bigl(t, \max\bigl\{\phi_n(t), \psi_n(t)\bigr\}\bigr)\Delta t$$
$$\leqslant \int_0^{\sigma^2(1)} G(x,t) f\bigl(t, \psi_n(t)\bigr)\Delta t = T\psi_n(x),$$

which implies

$$\phi_n(x) = T_n\phi_n(x) \leqslant T\psi_n(x), \quad x \in \bigl[0, \sigma^2(1)\bigr]_{\mathbb{T}},\ n \in \mathbb{N}. \tag{2.4}$$

Arguing as in Lemma 2.2 and using (2.4), it is fairly straightforward to show that there exists an $S > 0$ such that $\|\phi_n\| \leqslant S$ for all $n \in \mathbb{N}$. Similarly, we can follow the argument of Lemma 2.3 to show that there exists an $R > 0$ such that $\|\phi_n\| > R$ for all $n \in \mathbb{N}$.

For $\theta = pR$, (2.2) and the concavity of $\phi_n(x)$ for $x \in [0, \sigma^2(1)]_{\mathbb{T}}$ yields

$$g_\theta(x) \leqslant \phi_n(x), \quad x \in \bigl[0, \sigma^2(1)\bigr]_{\mathbb{T}}. \tag{2.5}$$

Therefore, the sequence $\{\phi_n\}_{n=1}^\infty$ is contained in the order interval $\langle g_\theta, S\rangle$, where S is the constant function of that value on $[0, \sigma^2(1)]_{\mathbb{T}}$; that is, $\{\phi_n\}_{n=1}^\infty \subset D$. Since $T: D \to D$ is a compact mapping, $T\phi_n \to \phi^*$ as $n \to \infty$ for some $\phi^* \in D$.

To conclude the proof of this theorem, we need to show that

$$\lim_{n\to\infty}\bigl(T\phi_n(x) - \phi_n(x)\bigr) = 0.$$

To that end, fix $\theta = pR$, and let $\varepsilon > 0$ be given. The latter part of Assumption (A1) permits us to choose $\delta \in (0, \sigma^2(1))_{\mathbb{T}}$ such that

$$\int_0^\delta f\bigl(t, g_\theta(t)\bigr)\Delta t < \frac{\varepsilon}{2M},$$

where

$$M = \max\bigl\{G(x,t)\colon (x,t) \in \bigl[0, \sigma^2(1)\bigr]_{\mathbb{T}} \times \bigl[0, \sigma(1)\bigr]_{\mathbb{T}}\bigr\}.$$

By (2.3) and (2.5), there exists an $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$,

$$\psi_n(t) \leqslant g_\theta(t) \leqslant \phi_n(t), \quad t \in \bigl[\delta, \sigma^2(1)\bigr]_{\mathbb{T}}.$$

Thus, for $t \in [\delta, \sigma^2(1)]_{\mathbb{T}}$,

$$f_n\bigl(t, \phi_n(t)\bigr) = f\bigl(t, \max\bigl\{\phi_n(t), \psi_n(t)\bigr\}\bigr) = f\bigl(t, \phi_n(t)\bigr),$$

and for $x \in [0, \sigma^2(1)]_{\mathbb{T}}$,

$$\begin{aligned}
T\phi_n(x)-\phi_n(x) &= T\phi_n(x)-T_n\phi_n(x)\\
&= \left[\int_0^{\delta} G(x,t)f\big(t,\phi_n(t)\big)\Delta t+\int_{\delta}^{\sigma^2(1)} G(x,t)f\big(t,\phi_n(t)\big)\Delta t\right]\\
&\quad -\left[\int_0^{\delta} G(x,t)f_n\big(t,\phi_n(t)\big)\Delta t+\int_{\delta}^{\sigma^2(1)} G(x,t)f_n\big(t,\phi_n(t)\big)\Delta t\right]\\
&= \int_0^{\delta} G(x,t)f\big(t,\phi_n(t)\big)\Delta t-\int_0^{\delta} G(x,t)f_n\big(t,\phi_n(t)\big)\Delta t.
\end{aligned}$$

Thus, for $x\in[0,\sigma^2(1)]_{\mathbb{T}}$,

$$\begin{aligned}
\big|T\phi_n(x)-\phi_n(x)\big| &\leqslant M\left[\int_0^{\delta} f\big(t,\phi_n(t)\big)\Delta t+\int_0^{\delta} f\big(t,\max\{\phi_n(t),\psi_n(t)\}\big)\Delta t\right]\\
&\leqslant M\left[\int_0^{\delta} f\big(t,\phi_n(t)\big)\Delta t+\int_0^{\delta} f\big(t,\phi_n(t)\big)\Delta t\right]\\
&\leqslant 2M\int_0^{\delta} f\big(t,g_\theta(t)\big)\Delta t<\varepsilon.
\end{aligned}$$

Since $x\in[0,\sigma^2(1)]_{\mathbb{T}}$ was arbitrary, we conclude that $\|T\phi_n-\phi_n\|<\varepsilon$ for all $n\geqslant n_0$. Hence, $\phi^*\in\langle g_\theta,S\rangle$ and for $x\in[0,\sigma^2(1)]_{\mathbb{T}}$,

$$T\phi^*(x)=T\Big(\lim_{n\to\infty}T\phi_n(x)\Big)=T\Big(\lim_{n\to\infty}\phi_n(x)\Big)=\lim_{n\to\infty}T\phi_n(x)=\phi^*(x). \qquad \square$$

3. The nabla–nabla problem

We now extend the existence results of the previous section to time scale boundary value problems of the form

$$y^{\nabla\nabla}+f(x,y)=0,\quad x\in(0,1]_{\mathbb{T}}, \tag{3.1}$$

$$y(\rho^2(0))=0,\quad y(p)=y(1), \tag{3.2}$$

where $f(x,y)$ is singular at $y=0$, and possibly at $x=0$ and $y=\infty$.

“Nabla–nabla” problems such as (3.1) are dealt with quite extensively in [8] as well as [6]. The paper by Eloe, Sheng, and Henderson [18] is a very interesting study in the numerical aspects of these equations.

Throughout this section, we make the following assumptions:

(B1) $p\in(0,1)_{\mathbb{T}}$ is arbitrary but fixed; $0,1\in\mathbb{T}$ with 0 right dense.

(B2) $f:(0,1]_{\mathbb{T}} \times (0,\infty)_{\mathbb{T}} \to (0,\infty)$ is decreasing in y for every $x \in (0,1]_{\mathbb{T}}$.
(B3) $\lim_{y\to 0^+} f(x,y)=\infty$ and $\lim_{y\to\infty} f(x,y)=0$, uniformly on compact subsets of $(\rho^2(0),1]_{\mathbb{T}}$.
(B4) $0<\int_{\rho^2(0)}^1 f(x,g_\theta(x))\nabla x<\infty$, for all $\theta>0$ where g_θ is defined in (3.3).

We seek positive solutions, $y:[\rho^2(0),1]_{\mathbb{T}} \to \mathbb{R}^+$, satisfying (3.1), (3.2). We note that positive solutions of (3.1), (3.2) are also concave on $[\rho^2(0),1]_{\mathbb{T}}$. Consider the Banach space $\mathcal{B}=C[\rho^2(0),1]_{\mathbb{T}}$ with the norm

$$\|u\| = \sup_{x\in[\rho^2(0),1]_{\mathbb{T}}} \left|u(x)\right|.$$

Define the normal cone, $\mathcal{K}\subset\mathcal{B}$, by

$$\mathcal{K} := \left\{u\in\mathcal{B} \mid u(x)\geqslant 0 \text{ on } \left[\rho^2(0),1\right]_{\mathbb{T}}\right\}.$$

Moreover, define the *tent function* $g_1:[\rho^2(0),1]_{\mathbb{T}}\to[0,\infty)$ by

$$g_1(x)=\begin{cases} x, & \text{if } \rho^2(0)\leqslant x\leqslant p,\\ p, & \text{if } \sigma(p)\leqslant x\leqslant 1,\end{cases}$$

and for $\theta>0$, let

$$g_\theta(x)=\theta\cdot g_1. \tag{3.3}$$

We observe that for each positive (and concave) solution, $y(x)$, of (3.1), (3.2), there exists some $\theta>0$ such that $g_\theta(x)\leqslant y(x)$ for all $x\in[\rho^2(0),1]_{\mathbb{T}}$.

We will apply Theorem 2.2 to operators whose kernel is the Green's function for $-y^{\nabla\nabla}=0$ and satisfies (3.2). This Green's function, $G:[\rho^2(0),1]_{\mathbb{T}}\times[\rho(0),1]_{\mathbb{T}}\to[0,\infty)$, is given by

$$G(x,t)=\begin{cases} x-\rho^2(0), & x\leqslant t\leqslant p,\\ \rho(t)-\rho^2(0), & \sigma(t)\leqslant x \text{ and } t\leqslant p,\\ \frac{(x-\rho^2(0))(1-\rho(t))}{1-p}, & \sigma(p)\leqslant t \text{ and } x\leqslant t,\\ \rho(t)-x+\frac{(x-\rho^2(0))(1-\rho(t))}{1-p}, & \sigma(p)\leqslant t\leqslant\sigma(t)\leqslant x.\end{cases}$$

Note that $G(x,t)>0$ for $(x,t)\in(\rho^2(0),1)_{\mathbb{T}}\times(\rho(0),1)_{\mathbb{T}}$.

Define $D\subset\mathcal{K}$ by

$$D := \left\{\phi\in\mathcal{K} \mid \exists\theta(\phi)>0 \text{ such that } \phi(x)\geqslant g_\theta(x),\ x\in\left[\rho^2(0),1\right]_{\mathbb{T}}\right\},$$

and the integral operator $T:D\to\mathcal{K}$ by

$$Tu(x) := \int_{\rho^2(0)}^{1} G(x,t)f\big(t,u(t)\big)\nabla t.$$

Using arguments very similar to the previous section, we obtain the analogous two lemmas establishing a priori upper and lower bounds on solutions as well as an existence theorem.

Lemma 3.1. *If f satisfies* (B1)–(B4), *then there exists an $S > 0$ such that $\|\phi\| \leqslant S$ for any solution $\phi \in D$ of* (3.1), (3.2).

Lemma 3.2. *If f satisfies* (B1)–(B4), *then there exists an $R > 0$ such that $\|\phi\| \geqslant R$ for any solution $\phi \in D$ of* (3.1), (3.2).

Theorem 3.1. *If f satisfies* (B1)–(B4), *then* (3.1), (3.2) *has at least one positive solution.*

4. The mixed delta–nabla and nabla–delta problems

Lastly, we extend these existence results to "mixed" time scales boundary value problems of the form

$$y^{\Delta\nabla} + f(x, y) = 0, \quad x \in (0, 1]_{\mathbb{T}}, \tag{4.1}$$

$$y\big(\rho(0)\big) = 0, \quad y(p) = y\big(\sigma(1)\big), \tag{4.2}$$

and

$$y^{\nabla\Delta} + f(x, y) = 0, \quad x \in (0, 1]_{\mathbb{T}}, \tag{4.3}$$

$$y\big(\rho(0)\big) = 0, \quad y(p) = y\big(\sigma(1)\big), \tag{4.4}$$

where $f(x, y)$ is singular at $y = 0$, and possibly at $x = 0$ and $y = \infty$.

"Delta–nabla" and "nabla–delta" problems such as (4.1), (4.3) are often referred to as mixed time scale boundary value problems. Various aspects of mixed problems have been investigated in the literature, such as Green's functions [9,11], the existence of multiple positive solutions [5], and the quasilinearization method [10]. Again, [18] is an excellent article on the numerical aspects of mixed time scales problems.

Once again, we make the following assumptions:

- (C1) $p \in (0, 1)_{\mathbb{T}}$ is arbitrary but fixed; $0, 1 \in \mathbb{T}$ with 0 right dense.
- (C2) $f : (0, 1]_{\mathbb{T}} \times (0, \infty)_{\mathbb{T}} \to (0, \infty)$ is decreasing in y for every $x \in (0, 1]_{\mathbb{T}}$.
- (C3) $\lim_{y\to 0^+} f(x, y) = \infty$ and $\lim_{y\to\infty} f(x, y) = 0$, uniformly on compact subsets of $(\rho(0), \sigma(1)]_{\mathbb{T}}$.
- (C4a) $0 < \int_{\rho(0)}^{\sigma(1)} f(x, g_\theta(x))\Delta x < \infty$, for all $\theta > 0$ for (4.1), (4.2) where g_θ is defined in (4.5).
- (C4b) $0 < \int_{\rho(0)}^{\sigma(1)} f(x, g_\theta(x))\nabla x < \infty$, for all $\theta > 0$ for (4.3), (4.4) where g_θ is defined in (4.5).

We seek positive solutions, $y : [\rho(0), \sigma(1)]_{\mathbb{T}} \to \mathbb{R}^+$, satisfying (4.1), (4.2) or (4.3), (4.4), respectively. Positive solutions of (4.1), (4.2) or (4.3), (4.4) are concave on $[\rho(0), \sigma(1)]_{\mathbb{T}}$. Consider the Banach space $\mathcal{B} = C[\rho(0), \sigma(1)]_{\mathbb{T}}$ for (4.1) and $\mathcal{B} = C[\rho(0), \sigma(1)]_{\mathbb{T}}$ for (4.3), each with the associated norm

$$\|u\| = \sup_{x\in[\rho(0),\sigma(1)]_{\mathbb{T}}} \big|u(x)\big|.$$

Define the normal cone, $\mathcal{K} \subset \mathcal{B}$, by

$$\mathcal{K} := \bigl\{u \in \mathcal{B} \mid u(x) \geqslant 0 \text{ on } \bigl[\rho(0), \sigma(1)\bigr]_{\mathbb{T}}\bigr\}.$$

Moreover, define the *tent function* $g_1 : [\rho(0), \sigma(1)]_{\mathbb{T}} \to [0, \infty)$ by

$$g_1(x) = \begin{cases} x, & \text{if } \rho(0) \leqslant x \leqslant p, \\ p, & \text{if } \sigma(p) \leqslant x \leqslant \sigma(1), \end{cases}$$

and for $\theta > 0$, let

$$g_\theta(x) = \theta \cdot g_1. \tag{4.5}$$

We observe that for each positive (and concave) solution, $y(x)$, of (4.1), (4.2) or (4.3), (4.4), there exists some $\theta > 0$ such that $g_\theta(x) \leqslant y(x)$ for all $x \in [\rho(0), \sigma(1)]_{\mathbb{T}}$.

We will apply Theorem 2.2 to operators whose kernel is the simultaneous Green's function for $-y^{\Delta\nabla} = 0$ and $-y^{\nabla\Delta} = 0$ which satisfies (4.2) or (4.4), respectively. This Green's function, $G : [\rho(0), \sigma(1)]_{\mathbb{T}} \times [0, 1]_{\mathbb{T}} \to [0, \infty)$, is given by

$$G(x,t) = \begin{cases} x - \rho(0), & x \leqslant t \leqslant p, \\ t - \rho(0), & \sigma(t) \leqslant x \text{ and } t \leqslant p, \\ \frac{(x-\rho(0))(\sigma(1)-t)}{\sigma(1)-p}, & \sigma(p) \leqslant t \text{ and } x \leqslant t, \\ t - x + \frac{(x-\rho(0))(\sigma(1)-t)}{\sigma(1)-p}, & \sigma(p) \leqslant t \leqslant \sigma(t) \leqslant x. \end{cases}$$

Note that $G(x,t) > 0$ for $(x,t) \in (\rho(0), \sigma(1))_{\mathbb{T}} \times (0,1)_{\mathbb{T}}$.

Define $D \subset \mathcal{K}$ by

$$D := \bigl\{\phi \in \mathcal{K} \mid \exists \theta(\phi) > 0 \text{ such that } \phi(x) \geqslant g_\theta(x),\ x \in \bigl[\rho(0), \sigma(1)\bigr]_{\mathbb{T}}\bigr\},$$

and the integral operators $T : D \to \mathcal{K}$ via

$$Tu(x) := \int_{\rho(0)}^{\sigma(1)} G(x,t) f\bigl(t, u(t)\bigr)\Delta t, \qquad \text{for (4.1), (4.2)},$$

$$Tu(x) := \int_{\rho(0)}^{\sigma(1)} G(x,t) f\bigl(t, u(t)\bigr)\nabla t, \qquad \text{for (4.3), (4.4)}.$$

Using arguments very similar to Section 2, we obtain the analogous two lemmas establishing a priori upper and lower bounds on solutions as well as an existence theorem.

Lemma 4.1. *If f satisfies* (C1)–(C4a,b)*, then there exists an $S > 0$ such that $\|\phi\| \leqslant S$ for any solution $\phi \in D$ of* (4.1)), (4.2) *and* (4.3), (4.4)*, respectively.*

Lemma 4.2. *If f satisfies* (C1)–(C4a,b)*, then there exists an $R > 0$ such that $\|\phi\| \geqslant R$ for any solution $\phi \in D$ of* (4.1), (4.2) *and* (4.3), (4.4)*, respectively.*

Theorem 4.1. *If f satisfies* (C1)–(C4a,b)*, then* (4.1), (4.2) *and* (4.3), (4.4) *have at least one positive solution, respectively.*

Appendix A. Time scales background

Since Stefan Hilger's 1988 PhD dissertation [26] which introduced analysis on time scales, there have been many publications relating difference equations with differential equations. A result for a dynamic equation contains simultaneously a corresponding result for a differential equation, one for a difference equation, as well as results for other dynamic equations in arbitrary time scales. The upshot here is *not* to reproduce blindly the vast body of similar theorems available for discrete and continuous dynamical systems, but rather to highlight both the similarities and (more often the case) the manifold differences in the two theories. Time scales theory presents us with the tools necessary to understand and explain the mathematical structure underpinning the theories of discrete and continuous dynamical systems and allows us to connect them. That is certainly the goal with this work. In fact, the potential impact of dynamic equations on time scales in applications is showcased in a recent cover story article in *New Scientist* magazine [31].

The following definitions and theorems (see Table 1), as well as a general introduction to the theory of dynamic equations on time scales, can be found in the excellent text by Bohner and Peterson [14].

A time scale $\mathbb{T}$ is any closed subset of $\mathbb{R}$. We define the forward and backward jump operators by $\sigma(t)=\inf\{s\in\mathbb{T}\colon s>t\}$ and $\rho(t)=\sup\{s\in\mathbb{T}\colon s<t\}$, respectively. An element $t\in\mathbb{T}$ is left-dense, right-dense, left-scattered, right-scattered if $\rho(t)=t$, $\sigma(t)=t$, $\rho(t)<t$, $\sigma(t)>t$, respectively. Also, $\inf\emptyset:=\sup\mathbb{T}$ and $\sup\emptyset:=\inf\mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_\kappa=\mathbb{T}-\{m\}$, otherwise $\mathbb{T}_\kappa=\mathbb{T}$ If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^\kappa=\mathbb{T}-\{M\}$, otherwise $\mathbb{T}^\kappa=\mathbb{T}$. The distance from an element $t\in\mathbb{T}$ to its successor is called the *graininess* of t and is denoted by $\mu(t)=\sigma(t)-t$. For $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^\kappa$, define $f^\Delta(t)$, the *delta derivative* of $f(t)$, as the number (when it exists), with the property that, for any $\varepsilon>0$, there exists a neighborhood U of t such that

$$\bigl|\bigl[f\bigl(\sigma(t)\bigr)-f(s)\bigr]-f^\Delta(t)\bigl[\sigma(t)-s\bigr]\bigr|\leqslant\varepsilon\bigl|\sigma(t)-s\bigr|,\quad \forall s\in U.$$

Note that the delta derivative is the usual derivative from calculus when $\mathbb{T}=\mathbb{R}$, but the delta derivative is the forward difference when $\mathbb{T}=\mathbb{Z}$. However, much more general time scales are possible.

A function $f:\mathbb{T}\to\mathbb{R}$ is *right dense continuous* (denoted $f\in C_{\mathrm{rd}}$) if it is continuous at every right dense point $t\in\mathbb{T}$, and its left hand limits exist at each left dense point $t\in\mathbb{T}$.

Table 1
Time scales: a generalization of discrete and continuous dynamics

$\mathbb{T}=\mathbb{R}$	$\mathbb{T}=\mathbb{Z}$	Any $\mathbb{T}$
Continuous ($\mu=0$)	Discrete ($\mu=1$)	Hybrid ($\mu=\mu(t)$)
ODE	forward difference equation	dynamic equation
$f'(t)$	$\Delta f(t)=f(t+1)-f(t)$	$f^\Delta(t)=\lim_{h\to\mu(t)}(f(t+h)-f(t))/h$
$(kf)'=k\cdot f'$	$\Delta(kf)=k\Delta f$	$(kf)^\Delta=k\cdot f^\Delta$
$(f+g)'=f'+g'$	$\Delta(f+g)=\Delta f+\Delta g$	$(f+g)^\Delta=f^\Delta+g^\Delta$
$(fg)'=fg'+f'g$	$\Delta(fg)=f\Delta g+\Delta f\cdot g(t+1)$	$(fg)^\Delta=f\cdot g^\Delta+f^\Delta\cdot g^\sigma$
$(f/g)'=\frac{f'g-fg'}{g^2}$	$\Delta(f/g)=\frac{\Delta f\cdot g-f\Delta g}{g\cdot g(t+1)}$	$(f/g)^\Delta=\frac{f^\Delta g-f\cdot g^\Delta}{g\cdot g^\sigma}$
$\int_a^b f(t)\,dt$	$\sum_{t=a}^{b-1}f(t),\ a<b$	$\int_a^b f(t)\,\Delta t$

We say f is delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is called the *delta derivative* of f on $\mathbb{T}^\kappa$. Lastly, a function $F : \mathbb{T} \to \mathbb{R}$ is called a *delta antiderivative* of $f : \mathbb{T}^\kappa \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. We can then define the (delta) definite integral of f by $\int_a^b f(\tau)\Delta\tau = F(b) - F(a)$.

The calculus of nabla derivatives is a generalization of the backward difference operator on $\mathbb{Z}$ to an arbitrary time scale. We refer the reader to Sections 8.3 and 8.4 of [14] for a detailed background on nabla derivatives.

A function $f : \mathbb{T} \to \mathbb{R}$ is *left dense continuous* (denoted $f \in C_{\mathrm{ld}}$) if it is continuous at every left dense point $t \in \mathbb{T}$, and its right hand limits exist at each right dense point $t \in \mathbb{T}$. We say f is nabla differentiable on $\mathbb{T}_\kappa$ provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$. The function $f^\nabla : \mathbb{T}_\kappa \to \mathbb{R}$ is called the *nabla derivative* of f on $\mathbb{T}_\kappa$. Lastly, a function $F : \mathbb{T} \to \mathbb{R}$ is called a *nabla antiderivative* of $f : \mathbb{T}_\kappa \to \mathbb{R}$ provided $F^\nabla(t) = f(t)$ holds for all $t \in \mathbb{T}_\kappa$. We can then define the (nabla) definite integral of f by $\int_a^b f(\tau)\nabla\tau = F(b) - F(a)$.

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